Energy - Conservative Forces, Linking W and U, Linking F and U

Note Title 9/29/2010

In AP Physics B, you learned that there are "conservative" and "nonconservative" forces:

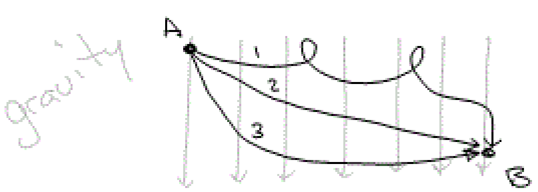
Conservative	Nonconservative
Gravity	Friction
Electrostatic	
Elastic	
Magnetic (more on this during the magnetism unit)	

Two Equivalent Definitions of "Conservative Force"

Definition 1

① A force which is "path-independent":

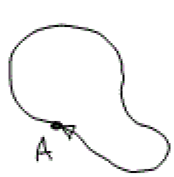
- The total work done when moving from point A to point B does not depend on the path taken:



$W_{AB_1} = W_{AB_2} = W_{AB_3}$

- Alternately, the total work done when returning to the starting point is zero:

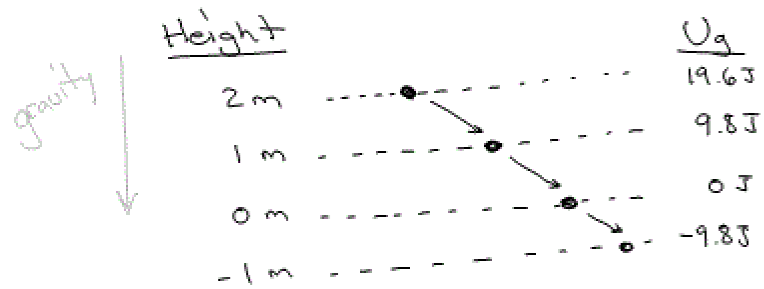
$W_{A \to A} = 0$



Definition 2

② A force for which a potential energy can be defined by an object's location.

Example: A 1 kg mass is in a gravitational field near Earth. $m = 1$ kg, $W = 9.8$ N ↓

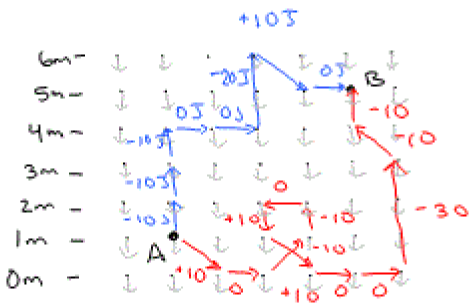


In the above example, the mass' potential energy is governed by its height.

→ Gravity is a conservative force!

Definition / Examples

Gravity – A conservative vector field
Calculate $W_{A \to B}$ along path ① and path ②
for a mass $m = 1.02\,kg$

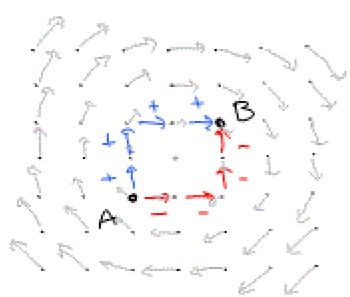


Each ↓ vector indicates a downward force of $mg = (1.02\,kg)(9.8\,\frac{m}{s^2})$
$mg = 10N$

For each segment $\Delta W = F \Delta d \cos\theta$

Path 1: $W_{TOTAL} = -10 - 10 - 10 + 0 + 0 - 20 + 10 + 0 = -40\,J$
Path 2: $W_{TOTAL} = -10 + 0 - 10 - 10 + 0 + 10 + 10 + 0 + 0 - 30 - 10 - 10 = -40\,J$
→ Work is the same for both paths!

A random non-conservative force vector field:



Path 1: $W_{TOTAL} = + + + +$
Path 2: $W_{TOTAL} = - - - -$
→ Work is different for both paths!

Special Note
Friction: Since friction always opposes motion, and $W_{FRICTION} = f \cdot d$, therefore work due to friction is extremely path-dependent (longer path = more work). By this definition, friction must be a nonconservative force!

Linking Work and Potential Energy

If a force is conservative according to Definition #2, work performed by that force must contribute to a loss in that form of potential energy. In other words:

Gravity	Electrostatic	Spring (Elastic)
$W_{gravity} = -\Delta U_g$	$W_{electrostatic} = -\Delta U_e$	$W_k = -\Delta U_k$

So if gravity causes an object to fall and does 10 J of work, that object will have lost 10 J of gravitational potential energy.

Linking Force and Potential Energy

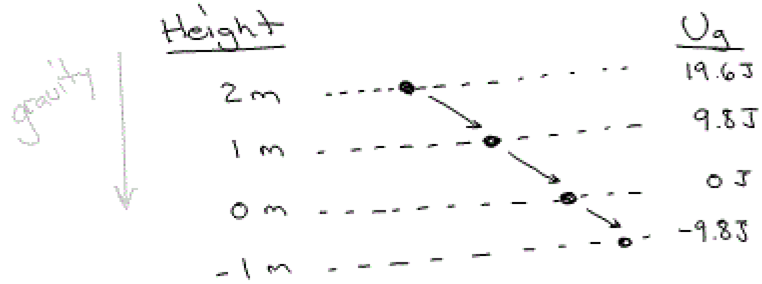
- The force on an object in a given field is given by the rate at which potential energy changes with respect to position in the field:

$$\vec{F} = -\frac{dU}{d\vec{r}}$$

Colloquially, the above relationship explains "stuff rolls downhill."

Also, "Things are lazy" → they always are forced in the direction of lowest energy.

Example: Given the gravitational potential energy scalar field shown below, calculate the force of gravity on an object whose mass $m = 1\text{ kg}$.



$$\vec{F} = -\frac{dU}{d\vec{r}} = -\frac{U_f - U_i}{r_f - r_i} = -\left(\frac{-9.8\text{ J} - 19.6\text{ J}}{-1\text{ m} - 2\text{ m}}\right)$$

$$= -\left(\frac{-29.4\text{ J}}{-3\text{ m}}\right)$$

$$= -9.8\ \frac{\text{J}}{\text{m}}$$

$$\boxed{F_{grav} = 9.8\text{ N downward}}$$

$\therefore$ Exactly as predicted by $F_{grav} = mg$!

Derivation of $F = -\frac{dU}{dr}$ for Gravity

Assuming an applied force F_{app} must do positive work W_{app} to lift an object up against gravity, the total work required is:

$$W_{app} = \int \vec{F}_{app} \cdot d\vec{r} = \Delta U_g$$

But a small increment of this work is just

$$dW_{app} = \vec{F}_{app} \cdot d\vec{r} = dU_g$$

And since $dW_{app} = -dW_{grav} = -\vec{F}_{grav} \cdot d\vec{r}$

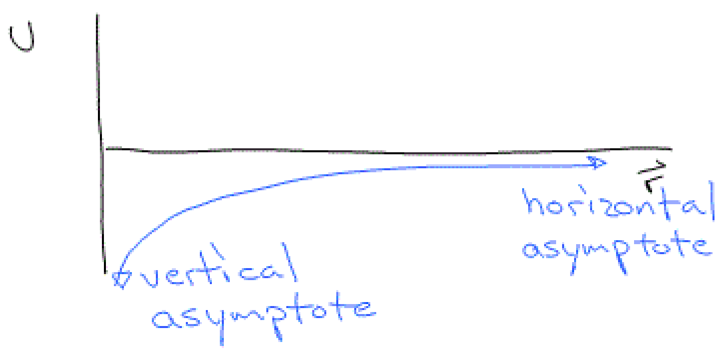
$$-\vec{F}_{grav} \cdot d\vec{r} = dU_g$$

$$\boxed{\vec{F}_{grav} = -\frac{dU_g}{d\vec{r}}}$$

Potential Energy Graphs: U vs r

- Since $\vec{F} = \frac{-dU}{d\vec{r}}$, there are a whole class of problems associated with calculating derivatives and graphing slopes.

Example: Create a $\vec{F}$ vs $\vec{r}$ graph from the following U vs $\vec{r}$ graph. The direction of positive force is toward the origin.



$$U = \frac{-Gm_1m_2}{\vec{r}}$$

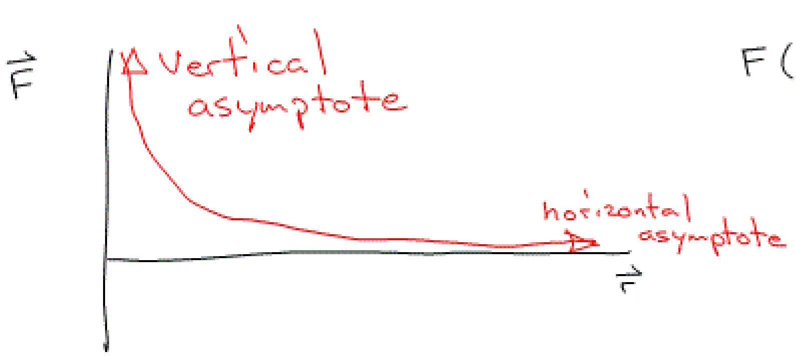
Solution:

$\vec{F} = -$ slope of U vs $\vec{r}$

$$F(r) = \frac{-dU}{dr} = \frac{-d}{dr}\left(Gm_1m_2r^{-1}\right)$$

$$F(r) = +\left(Gm_1m_2r^{-2}\right)$$

$$F(r) = \frac{Gm_1m_2}{r_2}$$



Law of Conservation of Energy

Together with charge, linear and angular momentum, and mass (which is just another form of energy), the Law of Conservation of Energy is one of the four apparently inviolate laws of the universe. No matter how hard we look, the following seems to hold true:

"For a closed,[1] isolated[2] system the total energy of the system never changes."

[1] closed = no mass transfer
[2] isolated = no heat transfer

Mathematically:

$$\sum E_0 = \sum E_1 = \sum E_2 = \sum E_3 = \sum E_4 = \ldots = \text{Always the same!}$$

For a system where no work is done on or by the system, the total mechanical energy is constant: $TME = K + U$

$$K_0 + U_0 = K_1 + U_1$$