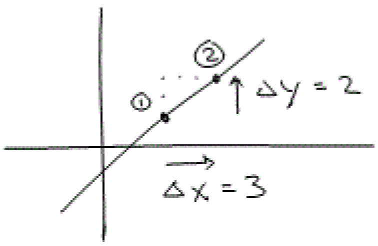
Math and Kinematics – Derivatives

Note Title 8/24/2010

1-Minute AP Calculus A Lecture:

Slope of a Line

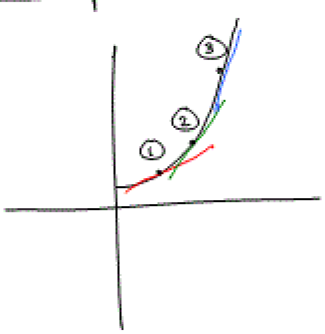


Average Slope Between ① and ②:

$$= \frac{\Delta y}{\Delta x} = \frac{2}{3}$$

For a straight line, slope is constant
→ slope is always $\frac{2}{3}$

Slope of a Curve



But how do we calculate a slope if the line is curved?

∴ The slope is changing!

∴ We would have to compute a slope at every point!

but... We can draw a tangent line at any point, then find the slope of it:

① Slope of red tangent
② Slope of green tangent
③ Slope of blue tangent

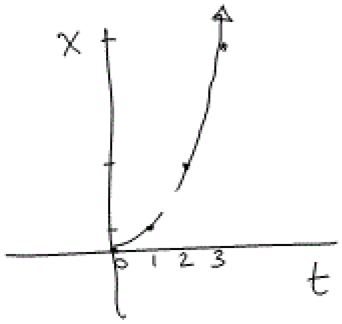
If we found the slopes at the three points, we would find that slope increases as x increases.

Example: If the equation for the curve above were $y = x^2 + 1$, and we computed slopes at $x = 1, 2, 3$, we would find:

x	slope
0	0
1	2
2	4
3	6

↓ increasing slope!

This is identical to what happens when you analyze a x vs t graph:

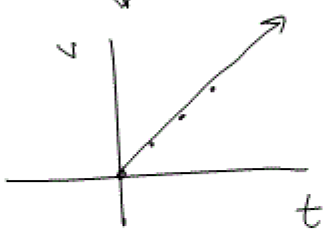


→ as t increases, x increases at an increasing rate

→ speeding up!

→ v increases!

↓ so



∴ v vs t increases!

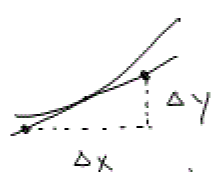
→ There must be some way to figure out the equation for the v vs t graph if we know the equation for the x vs t graph.

→ We call this "taking a derivative", or "differentiation".

Derivatives

If we want to find the slope of a tangent line:

$\text{slope} = \frac{\Delta y}{\Delta x}$ ← Difference between 2 points on the tangent line



This is pretty easy to do if you want to draw a picture for every problem, but what if I ask for the slope of $y = x^2 + 1$ when $x = 3$? You could graph it and estimate, but there is a quick way to find an exact answer...

(the answer is 6)

Derivatives - Power Rule

We can find an equation for the slope by applying the "power rule," which says

$$\underset{\substack{\text{for any}\\ x \text{ value}}}{\text{slope}} = \frac{dy}{dx} = \frac{d}{dx}\left(x^2+1\right) = 2x = \text{slope at any value of } x$$

then, just plug in $x = 3$, and you have slope $= 6$ when x is 3.

Power Rule: For a given term of a polynomial
1. Multiply the term by the exponent
2. Decrease the exponent by 1.

Examples:

Function $f(x)$		Derivative of Function $\frac{d}{dx}f(x)$
47	$\longrightarrow$	0 (the derivative of a constant is zero)
$x = x^1$	$\longrightarrow$	1
x^2	$\longrightarrow$	$2x$
x^3	$\longrightarrow$	$3x^2$
$\frac{1}{x} = x^{-1}$	$\longrightarrow$	$-x^{-2}$
$\sqrt{3x} = \sqrt{3}x^{\frac{1}{2}}$	$\longrightarrow$	$\frac{\sqrt{3}}{2}x^{-\frac{1}{2}}$
$\frac{-1}{\sqrt{x}} = -x^{-\frac{1}{2}}$	$\longrightarrow$	$\frac{1}{2}x^{-\frac{3}{2}}$

Derivatives: Distributive Property

The distributive property works for derivatives, just figure each term separately.

Example:
$$\frac{d}{dx}\left(2x^2+3x+8\right) = \frac{d}{dx}(2x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(8)$$
$$= 4x + 3 + 0$$
$$= 4x + 3$$

Derivatives: e^x

"e" (the "natural" number) is very special. It is the one number in nature for which the slope of a function is the original function:

$$\text{If } y = e^x$$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x$$

... but you still have to take into account any coefficients in front of x. So...

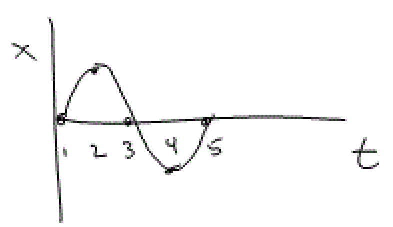
function	derivative of function
e^x $\longrightarrow$	e^x
e^{2x} $\longrightarrow$	$2e^{2x}$
$e^{-\frac{3}{2}x}$ $\longrightarrow$	$-\frac{3}{2}e^{-\frac{3}{2}x}$
$\frac{1}{2}e^{2x}$ $\longrightarrow$	e^{2x}
$-3e^{-\frac{1}{3}x}$ $\longrightarrow$	$e^{-\frac{1}{3}x}$

← Note that the exponent never changes!

Derivatives - Trig Functions

The slope of trig functions is just crazy...

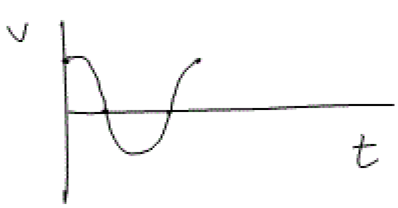
$x(t) = \sin(t)$



① + slope
② 0 slope
③ − slope
④ 0 slope
⑤ + slope

This looks like cosine!

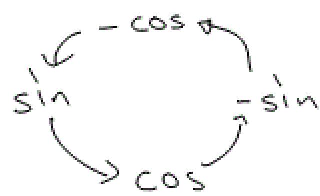
so...



→ a cosine function!

original function	derivative of function
$x(t) = A\sin(\omega t)$ $\longrightarrow$	$\frac{dx}{dt} = A\omega\cos(\omega t)$
$x(t) = A\cos(\omega t)$ $\longrightarrow$	$\frac{dx}{dt} = -A\omega\sin(\omega t)$

So the pattern is...



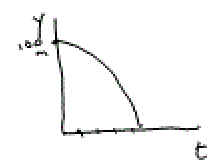
Derivatives - Kinematics Example

Example: The position of an object falling from the top of a 100 m tall cliff is given by the function:

$$y(t) = \tfrac{1}{2}a_y t^2 + y_0$$

or

$$y(t) = -4.9t^2 + 100$$

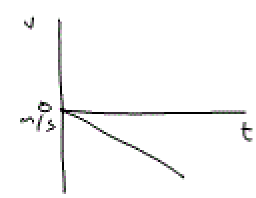


Find an expression for velocity ($v(t)$) and acceleration ($a(t)$).

velocity: $v(t) = \frac{d}{dt}x(t) = \frac{d}{dt}(-4.9t^2 + 100)$

$$= -9.8t + 0$$

$$\boxed{v(t) = -9.8t} \ (m/s)$$



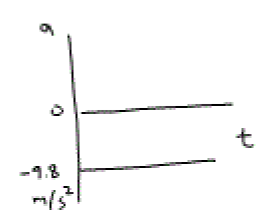
acceleration: $a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}(-9.8t)$

$$= -9.8$$

$$\boxed{a = -9.8 \ m/s^2}$$

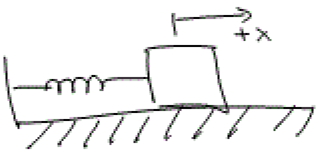
By the way, this is also referred to as the "second derivative" of x. This is written as $\frac{d^2x}{dt^2}$.

∴ Cool, huh???



Derivatives - SHM Example

The position of a spring-mass system is given by the function $x(t) = A\cos\left(\frac{2\pi}{T}t\right)$



A = amplitude
T = period

Find its acceleration as a function of time.

velocity: $v(t) = \frac{d}{dt}\left(A\cos\left(\frac{2\pi}{T}t\right)\right)$

$$v(t) = -\frac{2\pi A}{T}\sin\left(\frac{2\pi}{T}t\right)$$

acceleration: $a(t) = \frac{d}{dt}\left(-\frac{2\pi A}{T}\sin\left(\frac{2\pi}{T}t\right)\right)$

$$\boxed{a(t) = -\frac{4\pi^2}{T^2}A\cos\left(\frac{2\pi}{T}t\right)}$$