Energy – Work, Work-Energy, Dot Product, Line Integrals

Note Title 9/29/2010

Review from Last Year (Equations are written using AP notation)

Definition of Kinetic Energy $K = \frac{1}{2}mv^2$

Definition of Potential Energy (gravitational) $U_g = mgh$

Definition of work $W = Fd\cos\theta$

Work-Energy Theorem Net Work = Change in Kinetic Energy

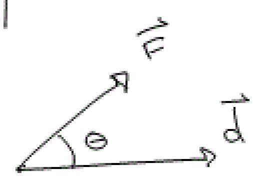
$W_{NET} = \Sigma W = \Delta K$

Definition of Mechanical Energy $TME = K + U$

Law of Conservation of Energy $\Sigma E_0 = \Sigma E_1 = \Sigma E_2 = \text{constant}$

WORK

- Last year's definition: (only good for constant force and straight line motion)



$W = Fd\cos\theta$

W = Work (scalar)
F = magnitude of force vector
d = magnitude of displacement vector
θ = angle between force and displacement vectors

But what if the force isn't constant?
Or if the path of travel is curved?
→ We have to take this into account!

Definition of Work for General (Curvilinear) Motion

$W = \int \vec{F} \cdot d\vec{r}$

$\vec{F}$ = force (vector)
$d\vec{r}$ = infinitesmal bit of displacement (vector

$\vec{F} \cdot d\vec{r}$ = "Dot Product" of $\vec{F}$ and $d\vec{r}$

$\int \vec{F} \cdot d\vec{r}$ = "Line Integral" of $\vec{F}$ and $d\vec{r}$

Dot Product (aka "Scalar Product")

- Last year, you learned how to: add scalars (2 apples + 2 apples = 4 apples)
  add vectors (2 m N + 2 m E = 2.8 m NE)
  multiply a scalar × vector
  (2 · 10 m/s up = 20 m/s up)

But you conveniently avoided multiplying two vectors.

- This year we are going to learn two different ways to multiply vectors:

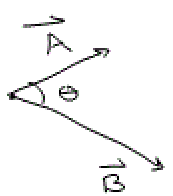
Dot Product: $\vec{A} \cdot \vec{B}$ = (some scalar)

Cross Product: $\vec{A} \times \vec{B}$ = (some other vector)

We'll hold off on the cross product until we discuss torque. For now, the Dot Product...

In Terms of Magnitude

$\vec{A} \cdot \vec{B} = AB \cos\theta$



In Terms of Components

$\vec{A} \cdot \vec{B} = A_i B_i + A_j B_j$

Dot product is:
Max when $\theta = 0°$
Zero when $\theta = 90°$
Max negative value when $\theta = 180°$

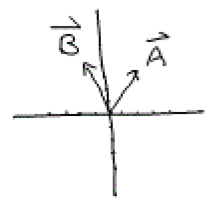
Cool, but probably not on AP test.

A cool feature of the dot product:

Find the angle in between the following vectors

$\vec{A} = 2\hat{i} + 3\hat{j}$

$\vec{B} = -1\hat{i} + 4\hat{j}$



$AB\cos\theta = A_i B_i + A_j B_j$

$(3.606)(3.162)\cos\theta = (2)(-1) + (3)(4)$

$11.40\cos\theta = -2 + 7$

$11.40\cos\theta = 5$

$\cos\theta = 0.438$

$\boxed{\theta = 64°}$ ⟶ 64° between $\vec{A}$ and $\vec{B}$!

$A = |\vec{A}| = \sqrt{2^2 + 3^2} = 3.606$

$B = |\vec{B}| = \sqrt{(-1)^2 + 4^2} = 3.162$

Back to the physics...

- Work is a Dot Product of $\vec{F}$ and displacement:

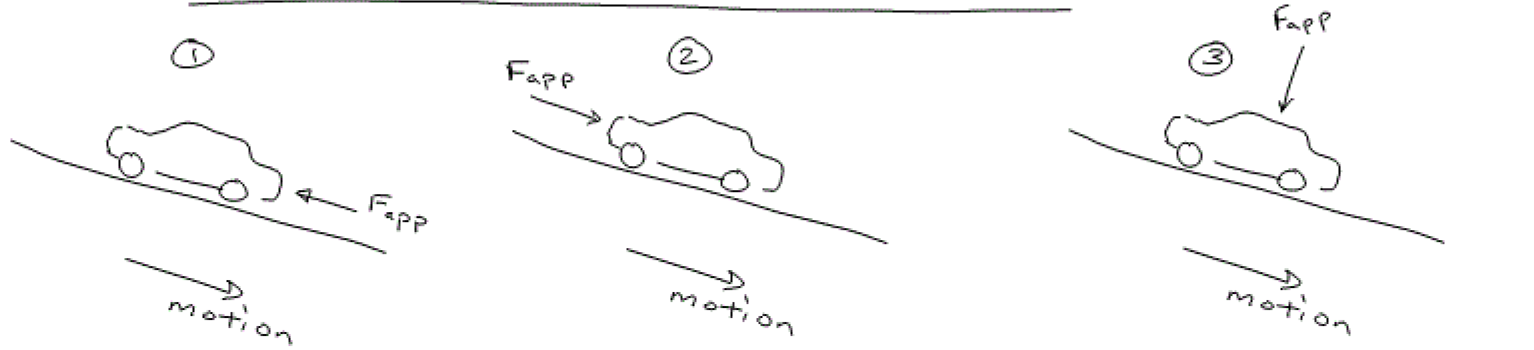
1. In other words, if a given force was in the same direction as the displacement, the force did positive work.
2. If $\vec{F}$ is perpendicular to displacement, zero work is done.
3. If $\vec{F}$ is opposed to the direction of motion, negative work is done.

Work-Energy Theorem

- The net work done on the object (the sum of the work done by all the forces on it) changes the kinetic energy of the object (not the potential energy).

$\boxed{W_{NET} = \Delta K}$ ⟵ This is slightly different than the version you learned last year!

3 Examples of a Car Rolling Downhill (Work-Energy Theorem)



Fapp does negative work, causing K to be reduced.

Fapp does positive work, causing K to be increased.

Fapp does **zero** work, therefore it has no effect on kinetic energy (K).

- In the 3 cases above, gravity is always pulling downhill
  → Wgravity is always positive!
- The net work is the sum of all the forces' work, so in case 1, if the car:

  speeds up → K↑ → gravity is doing more positive work than the applied force is doing negative work

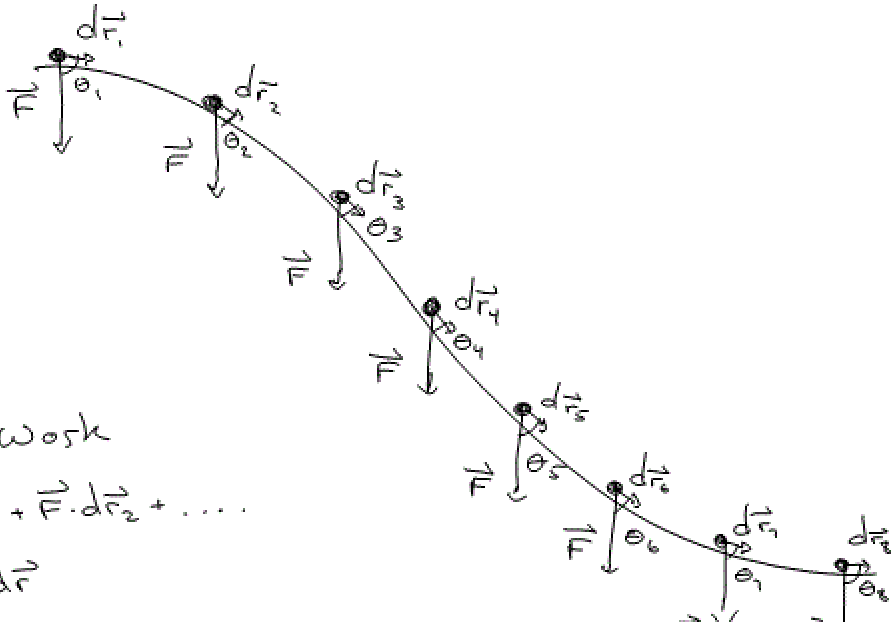
  slows down → K↓ → the applied force (+ friction) are doing more negative work than gravity is doing positive work

  rolls at constant speed → $\overleftrightarrow{K}$ → gravity work is matched by opposing work

Work as a Line Integral

- Last year's definition of work works great if the force is constant and the object is following a straight path. If the path is curved, however, we need to account for the work done during every infinitesmally small piece of the way:

Example: Find total work done on a rollercoaster by gravity as it rolls downhill.



Total work

$$W = \vec{F}\cdot d\vec{r}_1 + \vec{F}\cdot d\vec{r}_2 + \ldots$$

$$W = \Sigma \vec{F}\cdot d\vec{r}$$

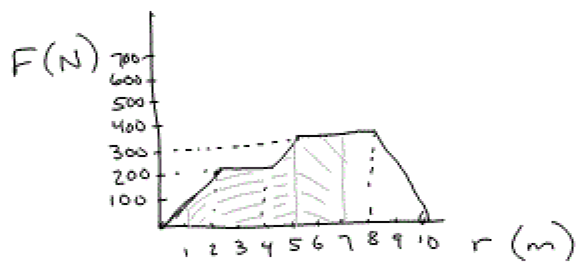
Of course, this is only an approximation. If we wanted an exact value, we would have to add up an infinite number of infinitesimal pieces from the start to finish:

$$W_{TOTAL} = \int \vec{F}\cdot d\vec{r}$$

This is the "Line Integral" of the dot product of $\vec{F}$ and $d\vec{r}$,

- Since work is an integral, it is also the "area under the $\vec{F}$ vs $\vec{r}$ curve."

Example: Find work done by a force that varies with position according to the following graph. The particle moves from $x = 1$ to $x = 7$.



$W_{1 \to 2m}$ = Area = 150 J

$2 \to 4m$ = 400 J

$4 \to 5m$ = 250 J

$5 \to 7m$ = 600 J

Total Work done by force = 1400 J