Math and Kinematics - Air Drag

Note Title 9/6/2010

Sliding Friction

Up until now, we have always treated friction force (F) as a constant that depends only on the normal force (N) and friction coefficient (μ).

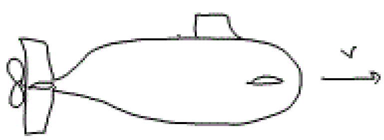
$$f = \mu N \quad \text{(sliding or static friction)}$$

But from real life experience, we know that in many cases, the force of friction depends on velocity. We call this type of friction "drag" to differentiate it from sliding friction:

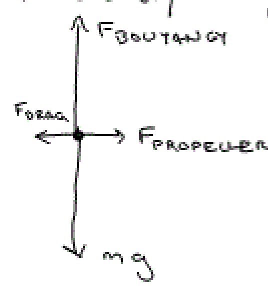
$$F_{DRAG} = bv$$

where F_{DRAG} = Drag force
v = velocity through a resistive medium
b = a constant

Example: Submarine moving through water at constant velocity:



Free Body Diagram



→ As submarine goes faster, drag force increases.
→ Greater force is required to push submarine through the water at faster speeds.

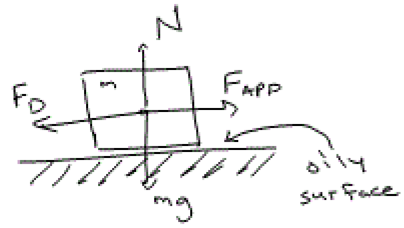
Drag Force Linearly Dependent on Velocity

Through experimentation, we have found that in many cases, drag force is "linearly dependent" on velocity. In other words, is speed doubles, drag doubles, and if speed triples, drag triples, etc. Mathematically, this "linearly dependent" relationship is:

$$F_{DRAG} = -bv$$

negative means drag is in direction opposite motion

a FBD of this situation might look like this:



A block sliding on an oily surface, where $F_D = bv$

Newton's 2nd Law for this situation would be

$\xrightarrow{+}$

$$F_{APP} - F_D = ma$$

$$F_{app} - bv = ma$$

But, since $a = \frac{dv}{dt}$ ...

$$\underbrace{F_{app} - bv = m\frac{dv}{dt}}_{\text{A differential equation}}$$

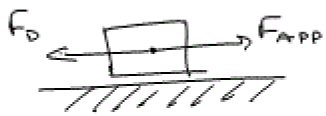
This is called a "differential equation" because it contains the differentials dv and dt. (Note that any equation expressed as a derivative is technically a differential equation, ie $\frac{dx}{dt} = 0$)

Solving Differential Equations

Differential Equations are great. I personally love them. they are fantastically easy to create from Newton's 2nd Law. The problem is: How do you use them to solve for position or velocity of an object as a function of time?

Constant Velocity - aka "steady state" - aka No Acceleration

If the block's speed is constant, $a = 0$.



So...

$$F_{APP} - bv = \cancel{ma}^{\,0}$$

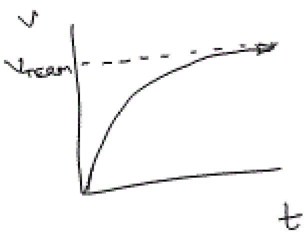
$$F_{APP} - bv = 0$$

$$F_{APP} = bv$$

$\therefore$ the "terminal velocity", or velocity reached at steady state is

$$\boxed{v_{TERM} = \frac{F_{APP}}{b}}$$

Solving for the velocity as a function of time prior to steady state



In reality, an object will asymptotically approach terminal velocity, and predicting its behavior prior to reaching terminal velocity requires us to use $ma \neq 0$:

$$F_{APP} - bv = ma$$

So, to find $v(t)$, we first replace $a = \frac{dv}{dt}$:

$$F_{app} - bv = m\frac{dv}{dt}$$

Trick #1

put $\frac{dv}{dt}$ by itself on left side

$$m\frac{dv}{dt} = F_{app} - bv$$

$$\frac{dv}{dt} = \frac{F_{APP} - bv}{m}$$

Trick #2

Divide all terms by "b" to get rid of coefficient in front of v (the integration variable)

$$\frac{dv}{dt} = \frac{\frac{F}{b} - \frac{b}{b}v}{\frac{m}{b}}$$

$$\frac{dv}{dt} = \frac{\frac{F}{b} - v}{\frac{m}{b}}$$

$$\frac{dv}{dt} = \frac{\frac{F}{b} - v}{\frac{m}{b}}$$

Trick #3

"Separate the Variables" - put v on left and t on right

$$\frac{dv}{\frac{F}{b} - v} = \frac{1}{\frac{m}{b}}\, dt$$

$$\frac{dv}{\frac{F}{b} - v} = \frac{b}{m}\, dt$$

Trick #4

multiply by -1 (you'll see why later)

$$\frac{dv}{v - F/b} = -\frac{b}{m}\, dt$$

Trick #5

Integrate both sides using $v(0) = 0$ and $t(0) = 0$ (assuming object started at rest)

$$\int_0^v \frac{1}{v - F/b}\, dv = \int_0^t -\frac{b}{m}\, dt$$

$$\ln\left(v - F/b\right)\Big|_0^v = -\frac{b}{m} \cdot t\,\Big|_0^t$$

$$\ln\left(v - F/b\right) - \ln\left(0 - F/b\right) = -\frac{b}{m} t$$

Trick #6

Use ln rules

$$\ln\left(\frac{v - F/b}{-F/b}\right) = -\frac{b}{m} t$$

$$\ln\left(-\frac{v}{F/b} + 1\right) = -\frac{b}{m} t$$

Trick #7

Get rid of ln!

$$-\frac{v}{F/b} + 1 = e^{-\frac{b}{m} t}$$

$$-\frac{v}{F/b} = e^{-\frac{b}{m} t} - 1$$

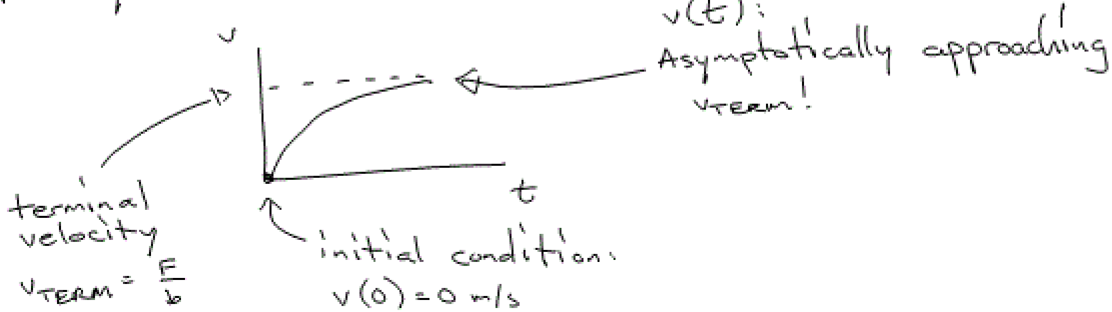
$$v = \frac{F}{b}\left(1 - e^{-\frac{b}{m} t}\right)$$

Trick #8

$\frac{F}{b}$ = Terminal Velocity!

$$\boxed{v(t) = v_{TERM}\left(1 - e^{-\frac{b}{m} t}\right)}$$

Graphically:



Solving for $x(t)$

since position is just the integral of velocity, and assuming that position starts at zero: $x(0) = 0$

$$x(t) = \int v\,dt = \int \frac{F}{b}\left(1 - e^{-\frac{b}{m}t}\right) dt$$

$$= \int \frac{F}{b}\,dt - \int e^{-\frac{b}{m}t}\,dt$$

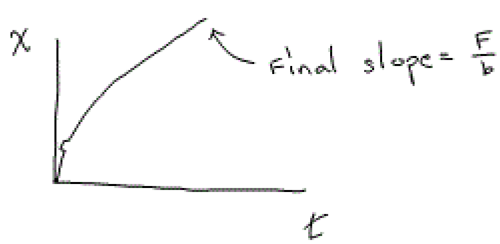
$$x(t) = \frac{F}{b}t + \frac{m}{b}e^{-\frac{b}{m}t} + C$$

$\leftarrow$ Using $x(0) = 0$

$$0 = 0 + \frac{m}{b}e^{0} + C$$

$$C = -\frac{m}{b}e^{0} = -\frac{m}{b}\cdot 1 = -\frac{m}{b}$$

$$x(t) = \frac{F}{b}t + \frac{m}{b}e^{-\frac{b}{m}t} - \frac{m}{b}$$



Solving for $a(t)$

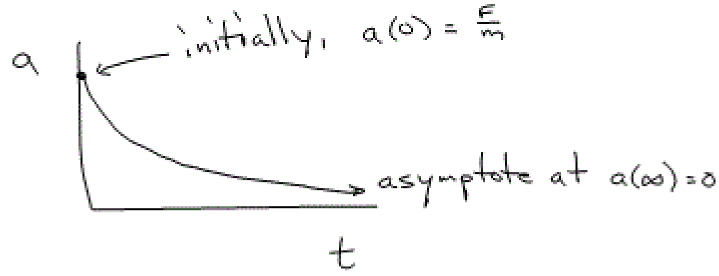
since $a = \frac{dv}{dt}$:

$$a = \frac{d}{dt}\left(\frac{F}{b}\left(1 - e^{-\frac{b}{m}t}\right)\right) = \frac{d}{dt}\left(\frac{F}{b}\right) - \frac{d}{dt}\left(\frac{F}{b}e^{-\frac{b}{m}t}\right)$$

$$a = 0 + \frac{F}{\cancel{b}}\frac{\cancel{b}}{m}e^{-\frac{b}{m}t}$$

$$a = \frac{F}{m}e^{-\frac{b}{m}t}$$

graphically:



Air Drag

Experiments have shown that air drag is not proportional to velocity, but to the square of velocity:

$$F_{\substack{AIR\\DRAG}} \propto v^2$$

More experiments have determined the relationship to be

$$F_{\substack{AIR\\DRAG}} = \frac{1}{2}\rho A c v^2$$

where

$F_{\substack{AIR\\DRAG}}$ = Drag force [Newtons]

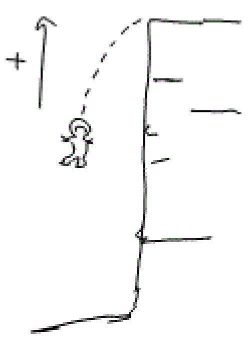
ρ = Air Density [kg/m^3]

A = Frontal cross-sectional area of the object [m^2]

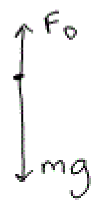
c = Drag coefficient which depends on the objects' shape [unitless]

v = Speed of object through the air [meters/second]

Example: A B.A.S.E. Jumper



FBD



Newton's 2nd Law

$$\Sigma F = ma$$

$$F_D - mg = ma$$

$$\frac{1}{2}\rho c A v^2 - mg = m\frac{dv}{dt}$$

solution (see handout)

$$\boxed{v(t) = v_T \frac{e^{-\sqrt{\frac{2g\rho Ac}{m}}\,t} - 1}{e^{-\sqrt{\frac{2g\rho Ac}{m}}\,t} + 1}}$$

Speed of a falling object

Terminal velocity $a = 0$

$$\Sigma F = \cancel{ma}^{\,0}$$

$$F_D = mg$$

$$v_T^2 = \frac{2mg}{\rho Ac}$$

$$v_T = \sqrt{\frac{2mg}{\rho Ac}}$$